On the geometric dependence of Riemannian Sobolev best constants *†

Ezequiel R. Barbosa, Marcos Montenegro ‡

Departamento de Matemática, Universidade Federal de Minas Gerais,

Caixa Postal 702, 30123-970, Belo Horizonte, MG, Brazil

Abstract

We concerns here with the continuity on the geometry of the second Riemannian L^p-Sobolev best constant $B_0(p,g)$ associated to the AB program. Precisely, for $1 \leq p \leq 2$, we prove that $B_0(p,g)$ depends continuously on g in the C^2-topology. Moreover, this topology is sharp for $p = 2$. From this discussion, we deduce some existence and C^0-compactness results on extremal functions.

1 Introduction and main results

Best constants and sharp first-order Sobolev inequalities on compact Riemannian manifolds have been extensively studied in the last few decades and surprising results have been obtained by showing the influence of the geometry on such problems. Particularly, the arising of concentration phenomena in PDEs has motivated the development of new methods in geometric analysis, see [2], [9] and [10] for a complete survey. Our interest here is the study of the behavior of the second Riemannian L^p-Sobolev best constant when the metric changes and some consequences such as existence and compactness results on extremal functions involving sets of Riemannian metrics.

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 2$. For $1 \leq p < n$, we denote by $H^p_1(M)$ the standard first-order Sobolev space defined as the completion of $C^\infty(M)$ with respect to the norm

*2000 Mathematics Subject Classification: 32Q10, 53C21

†Key words: Optimal Sobolev inequalities, second best constant, uniformity problem

‡*E-mail addresses*: ezequiel@mat.ufmg.br (E. R. Barbosa), montene@mat.ufmg.br (M. Montenegro)

$$||u||_{H_1^p(M)} = \left(\int_M |\nabla_g u|^p \, dv_g + \int_M |u|^p \, dv_g \right)^{1/p} ,$$

where dv_g denotes the Riemannian volume element of g. The Sobolev embedding theorem ensures that the inclusion $H_1^p(M) \subset L^{p^*}(M)$ is continuous for $p^* = \frac{np}{n-p}$. Thus, there exist constants $A, B \in \mathbb{R}$ such that for any $u \in H_1^p(M)$,

$$\left(\int_M |u|^{p^*} \, dv_g \right)^{p/p^*} \leq A \int_M |\nabla_g u|^p \, dv_g + B \int_M |u|^p \, dv_g \ . \qquad (I_g^p(A,B))$$

The first L^p-Sobolev best constant associated to $(I_g^p(A,B))$ is defined by

$$A_0(p,g) = \inf\{A \in \mathbb{R} : \ \text{there exists } \ B \in \mathbb{R} \ \text{such that} \ (I_g^p(A,B)) \ \text{is valid}\}$$

and, by Aubin [1], its value is given by

$$K(n,p)^p = \sup_{u \in \mathcal{D}_1^p(\mathbb{R}^n) \setminus \{0\}} \frac{\left(\int_{\mathbb{R}^n} |u|^{p^*} \, dx \right)^{p/p^*}}{\int_{\mathbb{R}^n} |\nabla u|^p \, dx}$$

and $\mathcal{D}_1^p(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm

$$||u||_{\mathcal{D}_1^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\nabla u|^p \, dx \right)^{1/p} \ .$$

In particular, the first best constant $A_0(p,g)$ does not depend on the geometry.

The first optimal Riemannain L^p-Sobolev inequality on $H_1^p(M)$ states that

$$\left(\int_M |u|^{p^*} \, dv_g \right)^{p/p^*} \leq K(n,p)^p \int_M |\nabla_g u|^p \, dv_g + B \int_M |u|^p \, dv_g \qquad (I_{g,opt}^p)$$

for some constant $B \in \mathbb{R}$. The validity of $(I_{g,opt}^p)$ has been established by Hebey and Vaugon [11] in the case $p = 2$, independently, by Aubin and Li [3] and Druet [6] in the case $1 < p < 2$, and by Druet [8] in the case $p = 1$.

For $1 \leq p \leq 2$, define then the second L^p-Sobolev best constant by

$$B_0(p,g) = \inf\{B \in \mathbb{R} : \ (I_{g,opt}^p) \ \text{is valid}\} \ .$$

Clearly, for any $u \in H_1^p(M)$, one has

$$\left(\int_M |u|^{p^*} \, dv_g\right)^{p/p^*} \leq K(n,p)^p \int_M |\nabla u|_g^p \, dv_g + B_0(p,g) \int_M |u|^p \, dv_g \ . \qquad (J_{g,opt}^p)$$

Note that $(J_{g,opt}^p)$ is sharp with respect to both the first and second best constants in the sense that none of them can be lowered. The inequality $(J_{g,opt}^p)$ is called the second optimal Riemannain L^p-Sobolev inequality.

On the contrary of the first best constant, the second one depends strongly on the geometry. In fact, note that $B_0(p, \lambda g) = \lambda^{-1} B_0(p,g)$ for any constant $\lambda > 0$. An interesting remark is that the arguments used in the works [3], [5], [6], [8] and [11] rely only on the continuity of derivatives up to second order of the components of g. Thus, a natural question is to know if $B_0(p,g)$ depends continuously on the metric g in the C^2-topology and if this topology is sharp.

Let M be a smooth compact manifold of dimension $n \geq 2$. Denote by $\mathcal{M}_2$ the space of smooth Riemannian metrics on M endowed with the C^2-topology and by $\mathcal{M}_\infty$ the space of smooth Riemannian metrics on M endowed with the usual Fréchet topology. We provide some answers for $1 \leq p \leq 2$ to the above question in the following theorems:

Theorem 1.1. *Let M be a smooth compact manifold of dimension n. If $n \geq 4$, then the map $g \in \mathcal{M}_2 \mapsto B_0(2,g)$ is continuous. Moreover, the C^2-topology is sharp.*

Theorem 1.2. *Let M be a smooth compact manifold of dimension n. If $n \geq 2$ and $1 \leq p < \min\{2, \sqrt{n}\}$, then the map $g \in \mathcal{M}_2 \mapsto B_0(p,g)$ is continuous.*

A direct consequence is:

Corollary 1.1. *Let M be a smooth compact manifold of dimension n. If either $1 \leq p \leq 2$ and $n \geq 4$ or $1 \leq p < \sqrt{n}$ and $n = 2, 3$, then the map $g \in \mathcal{M}_\infty \mapsto B_0(p,g)$ is continuous.*

The continuity question treated here is connected to the extremal functions C^0-compactness and a uniformity problem as follows. An extremal function of $(J_{g,opt}^p)$ is a nonzero function $u_0 \in H_1^p(M)$ such that

$$\left(\int_M |u_0|^{p^*} \, dv_g\right)^{p/p^*} = K(n,p)^p \int_M |\nabla_g u_0|^p \, dv_g + B_0(p,g) \int_M |u_0|^p \, dv_g \ .$$

Let $G \subset \mathcal{M}_2$. Set $E_p(G) = \bigcup_{g \in G} E_p(g)$, where $E_p(g)$ denotes the set of all extremal functions of $(J_{g,opt}^p)$ with unit L^{p^*}-norm.

Consider a subset $G \subset \mathcal{M}_2$ such that

$$B_0(2,g) > \frac{n-2}{4(n-1)} K(n,2)^2 \max_M Scal_g$$

for all metric $g \in G$. By Theorem 1 of [5], $E_2(g)$ is non-empty for all $g \in G$.

Theorem 1.1 then implies the following compactness result:

Corollary 1.2. *Let $n \geq 4$. If G is compact in the C^2-topology, then $E_2(G)$ is compact in the C^0-topology.*

Let $G \subset \mathcal{M}_2$. If $n \geq 2$ and $1 \leq p < \min\{2, \sqrt{n}\}$, by Theorem 2 of [5], $E_p(g)$ is non-empty and compact in the C^0-topology for all $g \in G$.

As a consequence of Theorem 1.2, we have:

Corollary 1.3. *Let $n \geq 2$ and $1 \leq p < \min\{2, \sqrt{n}\}$. If G is compact in the C^2-topology, then $E_p(G)$ is compact in the C^0-topology.*

Given a subset $G \subset \mathcal{M}_2$, the uniformity problem associated to $(I^p_{g,opt})$ consists in knowing if there exists a constant $B > 0$ such that for any $u \in H^p_1(M)$ and any $g \in G$,

$$\left(\int_M |u|^{p^*} \, dv_g \right)^{p/p^*} \leq K(n,p)^p \int_M |\nabla_g u|^p \, dv_g + B \int_M |u|^p \, dv_g \ . \qquad (I^p_g(G))$$

The existence of a such constant plays an important role in the study of Perelman's local non-collapsing properties along the Ricci flow. Recent advances in this direction have been obtained in [12], [15] and [16]. In this context, G represents the image of the flow in the space of metrics. The answer to this question clearly relies on properties of the set G. For example, as a consequence of Theorems 1.1 and 1.2, if either $1 \leq p \leq 2$ and $n \geq 4$ or $1 \leq p < \sqrt{n}$ and $n = 2, 3$, and G is compact in the C^2-topology, then a such constant $B > 0$ exists. In this case, we define

$$B_0(p,G) = \inf\{B \in \mathbb{R} : \ (I^p_g(G)) \ \text{ is valid for all } \ g \in G\} \ .$$

Clearly,

$$\left(\int_M |u|^{p^*} \, dv_g \right)^{p/p^*} \leq K(n,p)^p \int_M |\nabla_g u|^p \, dv_g + B_0(p,G) \int_M |u|^p \, dv_g \qquad (I^p_{g,opt}(G))$$

and

$$B_0(p,G) = \sup_{g \in G} B_0(p,g) \ .$$

Note that if $(I^p_{g,opt}(G))$ admits an extremal function for some metric $g \in G$, then $B_0(p,G) = B_0(p,g)$.

Existence results on extremal functions of $(I^p_{g,opt}(G))$ follow from results of [5] and from Theorems 1.1 and 1.2.

Precisely, we have:

Corollary 1.4. *Let $n \geq 4$ and $G \subset \mathcal{M}_2$ be such that*

$$B_0(2,g) > \frac{n-2}{4(n-1)} K^2(n,2) \max_M Scal_g$$

for all metric $g \in G$. If G is compact in the C^2-topology, then $(I^2_{g,opt}(G))$ admits at least an extremal function.

Corollary 1.5. *Let $n \geq 2$ and $1 \leq p < \min\{2, \sqrt{n}\}$. If G is compact in the C^2-topology, then $(I^p_{g,opt}(G))$ admits at least an extremal function.*

The proofs of Theorems 1.1 and 1.2 are made by contradiction. If the conclusions fail, we naturally are led to two possible alternatives. One of them is directly eliminated according to the definition of second best constant. The other alternative implies the existence of minimizers, concentrating in a point, of functionals associated to a family of metrics. The idea then consists in performing a concentration refined study on these minimizers in order to obtain the second contradiction. The proofs are inspired in the works [5], [6], [8] and [11]. New technical difficulties however arise when g changes in $\mathcal{M}_2$. In all the study of concentration, we assume only C^0-convergence of metrics. The C^2-convergence is necessary only in the last step of the proofs. For $p = 2$, we construct a counter-example by showing that the C^2-topology is sharp for the geometric continuity.

2 Proof of Theorem 1.1

Consider initially a sequence $(g_\alpha)_\alpha \subset \mathcal{M}_2$ converging to $g \in \mathcal{M}_2$ in the C^0-topology. The C^2-convergence will be used later in the last step of this proof. Suppose, by contradiction, that there exists $\varepsilon_0 > 0$ such that

$|B_0(2,g_\alpha) - B_0(2,g)| > \varepsilon_0$ for infinitely many α. Then, at least, one of the situations holds:

$$B_0(2,g) - B_0(2,g_\alpha) > \varepsilon_0 \ \text{ or } \ B_0(2,g_\alpha) - B_0(2,g) > \varepsilon_0$$

for infinitely many α. If the first alternative holds, then for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^{2^*} \, dv_{g_\alpha} \right)^{2/2^*} \leq K(n,2)^2 \int_M |\nabla_{g_\alpha} u|^2 \, dv_{g_\alpha} + (B_0(2,g) - \epsilon_0) \int_M u^2 \, dv_{g_\alpha} \ .$$

Letting $\alpha \to +\infty$ in this inequality, one finds

$$\left(\int_M |u|^{2^*} \, dv_g \right)^{2/2^*} \leq K(n,2)^2 \int_M |\nabla_g u|^2 \, dv_g + (B_0(2,g) - \varepsilon_0) \int_M u^2 \, dv_g,$$

and this clearly contradicts the definition of $B_0(2,g)$. Suppose then that the second situation holds, i.e. $B_0(2,g) + \varepsilon_0 < B_0(2,g_\alpha)$ for infinitely many α. For each $\alpha > 0$, consider the functional

$$J_\alpha(u) = \int_M |\nabla_{g_\alpha} u|^2 \, dv_{g_\alpha} + (B_0(2,g) + \varepsilon_0) K(n,2)^{-2} \int_M u^2 \, dv_{g_\alpha}$$

defined on the set $\Lambda_\alpha = \{u \in H_1^2(M) : \int_M |u|^{2^*} \, dv_{g_\alpha} = 1\}$. From the definition of $B_0(2,g_\alpha)$, one has

$$\lambda_\alpha := \inf_{u \in \Lambda_\alpha} J_\alpha(u) < K(n,2)^{-2} \ .$$

From this inequality, we find a nonnegative minimizer $u_\alpha \in \Lambda_\alpha$ for λ_α. The Euler-Lagrange equation for u_α is

$$-\Delta_{g_\alpha} u_\alpha + (B_0(2,g) + \varepsilon_0) K(n,2)^{-2} u_\alpha = \lambda_\alpha u_\alpha^{2^*-1}, \tag{E_α}$$

where $\Delta_{g_\alpha} u = \mathrm{div}_{g_\alpha}(\nabla_{g_\alpha} u)$ denotes the Laplace-Beltrami operator associated to the metric g_α. From the classical elliptic theory, it follows then that $u_\alpha \in C^\infty(M)$ and $u_\alpha > 0$ on M. Our goal now is to study the sequence $(u_\alpha)_\alpha$ as $\alpha \to +\infty$. Note first that

$$\int_M |\nabla_{g_\alpha} u_\alpha|^2 \, dv_{g_\alpha} + (B_0(2,g) + \varepsilon_0) K(n,2)^{-2} \int_M u_\alpha^2 \, dv_{g_\alpha} = \lambda_\alpha < K(n,2)^{-2}$$

and there exists a constant $c > 0$, independent of α, such that

$$\int_M u_\alpha^2 \, dv_g \le c \int_M u_\alpha^2 \, dv_{g_\alpha}$$

and

$$\int_M |\nabla_g u_\alpha|^2 \, dv_g \le c \int_M |\nabla_{g_\alpha} u_\alpha|^2 \, dv_{g_\alpha}$$

for $\alpha > 0$ large. Clearly, these inequalities imply that $(u_\alpha)_\alpha$ is bounded in $H_1^2(M)$. So, there exists $u \in H_1^2(M)$, $u \ge 0$, such that $u_\alpha \rightharpoonup u$ weakly in $H_1^2(M)$ and also $\lambda_\alpha \to \lambda$ as $\alpha \to +\infty$, with $0 \le \lambda \le K(n,2)^{-2}$, up to a subsequence. By the Sobolev embedding compactness theorem, we also have

$$\int_M u_\alpha^q \, dv_{g_\alpha} \to \int_M u^q \, dv_g$$

for any $1 \le q < 2^*$. Letting then $\alpha \to +\infty$ in the equation (E_α) and using that g_α converges to g in C^0, we find that u satisfies

$$-\Delta_g u + (B_0(2,g) + \varepsilon_0) K(n,2)^{-2} u = \lambda u^{2^*-1} \; . \tag{E}$$

If $u \not\equiv 0$, then $(J^2_{g,opt})$ and (E) imply

$$\left(\int_M u^{2^*} \, dv_g \right)^{2/2^*} < K(n,2)^2 \int_M |\nabla_g u|^2 \, dv_g + (B_0(2,g) + \varepsilon_0) \int_M u^2 \, dv_g$$

$$= K(n,2)^2 \lambda \int_M u^{2^*} \, dv_g \le \int_M u^{2^*} \, dv_g ,$$

so that $||u||_{2^*} > 1$. But, this is a contradiction, since

$$\int_M u^{2^*} dv_g \le \liminf_{\alpha \to +\infty} \int_M u_\alpha^{2^*} dv_{g_\alpha} = 1 \; .$$

We then assume that $u \equiv 0$ on M and will obtain another contradiction. In this case, we claim that $\lambda_\alpha \to K(n,2)^{-2}$ as $\alpha \to +\infty$. In fact, using that $u_\alpha \in \Lambda_\alpha$ and $g_\alpha \to g$ in C^0, one gets

$$\lim_{\alpha \to +\infty} \int_M u_\alpha^{2^*} \, dv_g = 1$$

and

$$\lim_{\alpha\to+\infty}\int_M u_\alpha^2\, dv_g = 0\ .$$

Letting $\alpha \to +\infty$ in

$$\left(\int_M u_\alpha^{2^*}\, dv_g\right)^{2/2^*} \le K(n,2)^2\int_M |\nabla_g u_\alpha|^2\, dv_g + B_0(2,g)\int_M u_\alpha^2\, dv_g$$

and using the limits above, one finds

$$\liminf_{\alpha\to+\infty}\int_M |\nabla_g u_\alpha|^2\, dv_g \ge K(n,2)^{-2}\ .$$

Clearly, the C^0-convergence of g_α then implies

$$\liminf_{\alpha\to+\infty}\int_M |\nabla_{g_\alpha} u_\alpha|^2\, dv_{g_\alpha} \ge K(n,2)^{-2}\ .$$

The claim follows then from

$$\limsup_{\alpha\to+\infty}\int_M |\nabla_{g_\alpha} u_\alpha|^2\, dv_{g_\alpha} \le \limsup_{\alpha\to+\infty}\lambda_\alpha \le K(n,2)^{-2}\ .$$

In the sequel, we divide the proof into six steps. Several possibly different positive constants, independent of α, will be denoted by c.

We say that $x \in M$ is a point of concentration of $(u_\alpha)_\alpha$ if, for any $\delta > 0$,

$$\limsup_{\alpha\to+\infty}\int_{B_g(x,\delta)} u_\alpha^{2^*}\, dv_{g_\alpha} > 0\ .$$

Step 1: The sequence $(u_\alpha)_\alpha$ possesses exactly one point of concentration x_0, up to a subsequence.

Proof: The existence of, at least, one point of concentration follows directly from the compactness of M, since $u_\alpha \in \Lambda_\alpha$. Conversely, let x_0 be a point of concentration of $(u_\alpha)_\alpha$. Let $\delta > 0$ small and consider a smooth function $\eta \in C_0^\infty(B_g(x_0,\delta))$ such that $0 \le \eta \le 1$ and $\eta = 1$ in $B_g(x_0,\delta/2)$. Multiplying (E_α) by $\eta^2 u_\alpha^k$, $k > 1$, and integrating over M, one has

$$-\int_M \eta^2 u_\alpha^k \Delta_{g_\alpha} u_\alpha \, dv_{g_\alpha} + (B_0(2,g)+\varepsilon_0) K(n,2)^{-2} \int_M \eta^2 u_\alpha^{k+1} \, dv_{g_\alpha} = \lambda_\alpha \int_M \eta^2 u_\alpha^{k+2^*-1} \, dv_{g_\alpha} \, . \tag{1}$$

For each $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$, independent of α, since $g_\alpha \to g$ in C^0, such that

$$\int_M |\nabla_{g_\alpha}(\eta u_\alpha^{\frac{k+1}{2}})|^2 \, dv_{g_\alpha} \le \frac{(k+1)^2}{4}(1+\varepsilon) \int_M \eta^2 u_\alpha^{k-1} |\nabla_{g_\alpha} u_\alpha|^2 \, dv_{g_\alpha}$$

$$+ c_\varepsilon ||\nabla \eta||_\infty^2 \int_M u_\alpha^{k+1} \, dv_{g_\alpha}$$

for $\alpha > 0$ large. By direct integration, we have

$$-\int_M \eta^2 u_\alpha^k \Delta_{g_\alpha} u_\alpha \, dv_{g_\alpha} \ge k \int_M \eta^2 u_\alpha^{k-1} |\nabla_{g_\alpha} u_\alpha|^2 \, dv_{g_\alpha} - \int_M u_\alpha^k |\nabla_{g_\alpha} u_\alpha| |\nabla_{g_\alpha}(\eta^2)| \, dv_{g_\alpha},$$

so that, together with (1),

$$\int_M |\nabla_{g_\alpha}(\eta u_\alpha^{\frac{k+1}{2}})|^2 \, dv_{g_\alpha} \le \frac{(k+1)^2}{4k}(1+\varepsilon)\lambda_\alpha \int_M \eta^2 u_\alpha^{k+2^*-1} \, dv_{g_\alpha} \tag{2}$$

$$+ \frac{(k+1)^2}{4k}(1+\varepsilon) \int_M u_\alpha^k |\nabla_{g_\alpha} u_\alpha| |\nabla_{g_\alpha}(\eta^2)| \, dv_{g_\alpha} + c_\varepsilon ||\nabla \eta||_\infty^2 \int_M u_\alpha^{k+1} \, dv_{g_\alpha} \, .$$

From the Hölder inequality, one has

$$\int_M \eta^2 u_\alpha^{k+2^*-1} \, dv_{g_\alpha} \le \left(\int_M (\eta u_\alpha^{\frac{k+1}{2}})^{2^*} \, dv_{g_\alpha} \right)^{2/2^*} \left(\int_{B_g(x_0,\delta)} u_\alpha^{2^*} \, dv_{g_\alpha} \right)^{1-2/2^*} \tag{3}$$

and

$$\int_M u_\alpha^k |\nabla_{g_\alpha} u_\alpha| |\nabla_{g_\alpha}(\eta^2)| \, dv_{g_\alpha} \le 2||\nabla \eta||_\infty \left(\int_M |\nabla_{g_\alpha} u_\alpha|^2 \, dv_{g_\alpha} \right)^{1/2} \left(\int_M u_\alpha^{2k} \, dv_{g_\alpha} \right)^{1/2} \, . \tag{4}$$

For each $\varepsilon > 0$, there exists a constant $d_\varepsilon > 0$, independent of α, such that

$$\left(\int_M (\eta u_\alpha^{\frac{k+1}{2}})^{2^*} \, dv_{g_\alpha} \right)^{2/2^*} \le (K(n,2)^2 + \varepsilon) \int_M |\nabla_{g_\alpha}(\eta u_\alpha^{\frac{k+1}{2}})|^2 \, dv_{g_\alpha} + d_\varepsilon \int_M u_\alpha^{k+1} \, dv_{g_\alpha} \tag{5}$$

for $\alpha > 0$ large. Here is used that $(1-\varepsilon)g \le g_\alpha \le (1+\varepsilon)g$ in the bilinear forms sense. From $J_\alpha(u_\alpha) < K(n,2)^{-2}$, one has

$$\left(\int_M |\nabla_{g_\alpha} u_\alpha|^2 \, dv_{g_\alpha}\right)^{1/2} \le \left(\lambda_\alpha \int_M u_\alpha^{2^*} \, dv_{g_\alpha}\right)^{1/2} \le K(n,2)^{-1} \; . \tag{6}$$

So, putting together (2), (3), (4), (5) and (6), one finds

$$A_\alpha \left(\int_M (\eta u_\alpha^{\frac{k+1}{2}})^{2^*} \, dv_{g_\alpha}\right)^{2/2^*} \le B \int_M u_\alpha^{k+1} \, dv_{g_\alpha} + C \left(\int_M u_\alpha^{2k} \, dv_{g_\alpha}\right)^{1/2} , \tag{7}$$

where

$$A_\alpha = 1 - \frac{(k+1)^2}{4k}(1+\varepsilon)^2 \lambda_\alpha K(n,2)^2 \left(\int_{B_g(x_0,\delta)} u_\alpha^{2^*} \, dv_{g_\alpha}\right)^{1-2/2^*} ,$$

$$B = K(n,2)^2(1+\varepsilon)c_\varepsilon ||\nabla \eta||_\infty^2 + d_\varepsilon$$

and

$$C = 2\frac{(k+1)^2}{4k}(1+\varepsilon)^2 ||\nabla\eta||_\infty K(n,2) \; .$$

Since x_0 is a point of concentration of $(u_\alpha)_\alpha$, we have

$$\limsup_{\alpha \to +\infty} \left(\int_{B_g(x_0,\delta)} u_\alpha^{2^*} \, dv_{g_\alpha}\right)^{1-2/2^*} = a > 0,$$

with $a \le 1$, since $u_\alpha \in \Lambda_\alpha$. We claim that $a = 1$ for all $\delta > 0$. In fact, if $a < 1$ for some $\delta > 0$, taking $\varepsilon > 0$ small enough and $k > 1$ close to 1 such that $A_\alpha > A$, where A is a positive constant and independent of α. Since the right-hand side of (7) is bounded for k close to 1, we find a constant $c > 0$, independent of α, such that

$$\left(\int_M (\eta u_\alpha^{\frac{k+1}{2}})^{2^*} \, dv_{g_\alpha}\right)^{2/2^*} \le c$$

for $\alpha > 0$ large. From the Hölder inequality, one has

$$\int_{B_g(x_0,\frac{\delta}{2})} u_\alpha^{2^*} \, dv_{g_\alpha} = \int_{B_g(x_0,\frac{\delta}{2})} u_\alpha^{k+1} \, u_\alpha^{2^*-1-k} \, dv_{g_\alpha}$$

$$\leq \left(\int_M (\eta u_\alpha^{\frac{k+1}{2}})^{2^*} \, dv_{g_\alpha} \right)^{2/2^*} \left(\int_M u_\alpha^{2^* - \frac{2^*(k-1)}{2^*-2}} \, dv_{g_\alpha} \right)^{1-2/2^*} \leq c \left(\int_M u_\alpha^{2^* - \frac{2^*(k-1)}{2^*-2}} \, dv_{g_\alpha} \right)^{1-2/2^*} .$$

Choose k close to 1 such that $2 < 2^* - \frac{2^*(k-1)}{2^*-2} < 2^*$. Since $||u_\alpha||_2 \to 0$, it follows then from an interpolation argument that

$$\limsup_{\alpha \to +\infty} \int_{B_g(x_0, \frac{\delta}{2})} u_\alpha^{2^*} \, dv_{g_\alpha} = 0 \ .$$

But this clearly contradicts the fact that x_0 is a point of concentration. Therefore, $a = 1$ and

$$\limsup_{\alpha \to +\infty} \int_{B_g(x_0, \delta)} u_\alpha^{2^*} \, dv_{g_\alpha} = 1$$

for all $\delta > 0$. Since $u_\alpha \in \Lambda_\alpha$, it follows then that $(u_\alpha)_\alpha$ has exactly one point of concentration, up to a subsequence. ∎

Step 2: Let $x_0 \in M$ be the unique point of concentration of $(u_\alpha)_\alpha$. Then,

$$\lim_{\alpha \to +\infty} u_\alpha = 0 \ \text{ in } \ C^0_{loc}(M \setminus \{x_0\}) \ . \tag{8}$$

Proof: From (7), given $\overline{\Omega} \subset M \setminus \{x_0\}$, there exist constants $\varepsilon, c_1 > 0$, independent of α, such that

$$\int_\Omega u_\alpha^{2^*(1+\varepsilon)} \, dv_{g_\alpha} \leq c_1$$

for $\alpha > 0$ large. On the other hand, from the C^0-convergence of g_α, we find constants γ and c_0 such that $g_\alpha \geq \gamma \xi$, in the bilinear forms sense, and $||(g_\alpha)_{ij}||_{C^0} \leq c_0$ for $\alpha > 0$ large, where ξ stands for the Euclidean metric on $\mathbb{R}^n$. Finally, the conclusion (8) follows from a De Giorgi-Nash-Moser iterative scheme applied to (E_α). Here, it is important to note that the involved constants in this scheme depend only on γ, c_0 and c_1. We refer for instance to Serrin [14] for more details. ∎

Let $x_\alpha \in M$ be a maximum point of u_α, i.e. $u_\alpha(x_\alpha) = ||u_\alpha||_\infty$. By the steps 1 and 2, one has $x_\alpha \to x_0$ as $\alpha \to +\infty$.

Step 3: For each $R > 0$, we have

$$\lim_{\alpha\to+\infty}\int_{B_{g_\alpha}(x_\alpha,R\mu_\alpha)} u_\alpha^{2^*}\, dv_{g_\alpha} = 1 - \varepsilon_R \tag{9}$$

where $\mu_\alpha = ||u_\alpha||_\infty^{-2^*/n}$ and $\varepsilon = \varepsilon_R \to 0$ as $R \to +\infty$.

Proof: From

$$1 = \int_M u_\alpha^{2^*}\, dv_{g_\alpha} \le ||u_\alpha||_\infty^{2^*-2}\int_M u_\alpha^2\, dv_{g_\alpha},$$

we find $||u_\alpha||_\infty \to +\infty$ as $\alpha \to +\infty$, since $\int_M u_\alpha^2\, dv_{g_\alpha} \to 0$. So, $\mu_\alpha \to 0$ as $\alpha \to +\infty$. Let $\exp_{x_\alpha}$ be the exponential map at x_α with respect to the metric g. Since $x_\alpha \to x_0$, there exists $\delta > 0$, independent of α, such that $\exp_{x_\alpha}$ map $B(0,\delta) \subset \mathbb{R}^n$ onto $B_g(x_\alpha,\delta)$ for $\alpha > 0$ large. For each $x \in B(0,\delta\mu_\alpha^{-1})$, set

$$\tilde{g}_\alpha(x) = (\exp_{x_\alpha}^* g_\alpha)(\mu_\alpha x)$$

and

$$\varphi_\alpha(x) = \mu_\alpha^{n/2^*} u_\alpha(\exp_{x_\alpha}(\mu_\alpha x)) \ .$$

As one easily checks,

$$-\Delta_{\tilde{g}_\alpha}\varphi_\alpha + (B_0(2,g)+\varepsilon_0)K(n,2)^{-2}\mu_\alpha^2\varphi_\alpha = \lambda_\alpha\varphi_\alpha^{2^*-1} \ . \tag{$\tilde{E}_\alpha$}$$

Clearly,

$$\tilde{g}_\alpha \to \xi \ \text{ in } \ C^0_{loc}(\mathbb{R}^n) \ . \tag{10}$$

In particular, for each bounded open $\Omega \subset \mathbb{R}^n$, there exist constants $\gamma, c_0 > 0$ such that

$$\tilde{g}_\alpha \ge \gamma\xi \ \text{ in } \ \Omega, \tag{11}$$

in the bilinear forms sense, and

$$||(\tilde{g}_\alpha)_{ij}||_{C^0(\Omega)} \le c_0 \tag{12}$$

for $\alpha > 0$ large. So, from (11), there exists a constant $c > 0$ such that

$$\int_\Omega |\nabla \varphi_\alpha|^2 \, dv_\xi \le c \int_{B(0,\delta\mu_\alpha^{-1})} |\nabla_{\tilde{g}_\alpha} \varphi_\alpha|^2 \, dv_{\tilde{g}_\alpha} = c \int_{B(x_\alpha,\delta)} |\nabla_{g_\alpha} u_\alpha|^2 \, dv_{g_\alpha} \le cK(n,2)^{-2}$$

and

$$\int_\Omega \varphi_\alpha^{2^*} \, dv_\xi \le c \int_{B(0,\delta\mu_\alpha^{-1})} \varphi_\alpha^{2^*} \, dv_{\tilde{g}_\alpha} = c \int_{B(x_\alpha,\delta)} u_\alpha^{2^*} \, dv_{g_\alpha} \le c \; .$$

Therefore, the sequence $(\varphi_\alpha)_\alpha$, with $\alpha > 0$ large, is bounded in $H_1^2(\Omega)$ for any bounded open $\Omega \subset \mathbb{R}^n$, so that $\varphi_\alpha \rightharpoonup \varphi$ weakly in $H_1^2(\Omega)$, $\varphi \ge 0$, and $\int_\Omega \varphi_\alpha^q \, dv_\xi \to \int_\Omega \varphi^q \, dv_\xi$ for any $1 \le q < 2^*$, up to a subsequence. Then, letting $\alpha \to +\infty$ in $(\tilde{E}_\alpha)$, using (10), $\lambda_\alpha \to K(n,2)^{-2}$ and $\mu_\alpha \to 0$, we conclude that φ satisfies in the weak sense,

$$-\Delta \varphi = K(n,2)^{-2} \varphi^{2^*-1} \text{ in } \mathbb{R}^n \; . \tag{13}$$

Note also that $\varphi \in \mathcal{D}_1^2(\mathbb{R}^n)$. This last fact follows directly from

$$\int_\Omega |\nabla \varphi_\alpha|^2 \, dv_\xi \le cK(n,2)^{-2}$$

and $\varphi_\alpha \rightharpoonup \varphi$ in $H_1^2(\Omega)$. Thanks to (11), (12) and the bound of $(\varphi_\alpha)_\alpha$ and $(\mu_\alpha)_\alpha$, classical Hölder estimates on elliptic PDEs weak solutions (see [13]) can be applied to $(\tilde{E}_\alpha)$, so that $(\varphi_\alpha)_\alpha$ is uniformly bounded in $C^\beta(\overline{\Omega})$ for any bounded open $\Omega \subset \mathbb{R}^n$ and $\alpha > 0$ large. Therefore, $\varphi_\alpha \to \varphi$ in $C^0_{loc}(\mathbb{R}^n)$, up to a subsequence, so that $\varphi \not\equiv 0$, since $\varphi_\alpha(0) = 1$ for all α. From the equation (13), one has

$$\int_{\mathbb{R}^n} |\nabla \varphi|^2 \, dv_\xi = K(n,2)^{-2} \int_{\mathbb{R}^n} \varphi^{2^*} \, dv_\xi,$$

since $\varphi \in \mathcal{D}_1^2(\mathbb{R}^n)$. So,

$$K(n,2)^{-2} \left(\int_{\mathbb{R}^n} \varphi^{2^*} \, dv_\xi \right)^{2/2^*} \le \int_{\mathbb{R}^n} |\nabla \varphi|^2 \, dv_\xi = K(n,2)^{-2} \int_{\mathbb{R}^n} \varphi^{2^*} \, dv_\xi,$$

so that

$$\int_{\mathbb{R}^n} \varphi^{2^*}\, dv_\xi \geq 1\ .$$

On the other hand, since

$$\int_\Omega \varphi_\alpha^{2^*}\, dv_{\tilde{g}_\alpha} \leq \int_{B(0,\delta\mu_\alpha^{-1})} \varphi_\alpha^{2^*}\, dv_{\tilde{g}_\alpha} = \int_{B_{g_\alpha}(x_\alpha,\delta)} u_\alpha^{2^*}\, dv_{g_\alpha} \leq 1,$$

we find $\int_{\mathbb{R}^n} \varphi^{2^*}\, dv_\xi = 1$, so that the conclusion of this step follows from the convergence

$$\int_{B_{g_\alpha}(x_\alpha,R\mu_\alpha)} u_\alpha^{2^*}\, dv_{g_\alpha} = \int_{B(0,R)} \varphi_\alpha^{2^*}\, dv_{\tilde{g}_\alpha} \to \int_{B(0,R)} \varphi^{2^*}\, dv_\xi\ . \quad \blacksquare$$

Step 4: There exists a constant $c > 0$, independent of α, such that

$$d_g(x, x_\alpha)^{n/2^*} u_\alpha(x) \leq c$$

for all $x \in M$ and α large, where d_g stands for the distance with respect to the metric g.

Proof: Set $\omega_\alpha(x) = d_g(x, x_\alpha)^{n/2^*} u_\alpha(x)$ for $x \in M$ and suppose, by contradiction, that the conclusion of this step fails. In this case,

$$\lim_{\alpha\to+\infty} ||\omega_\alpha||_\infty = +\infty,$$

up to a subsequence. We next will derive a contradiction. Let $y_\alpha \in M$ be a maximum point of ω_α. Note that $u_\alpha(y_\alpha) \to +\infty$ and

$$\lim_{\alpha\to+\infty} \frac{d_g(y_\alpha, x_\alpha)}{\mu_\alpha} = +\infty, \tag{14}$$

since

$$\frac{d_g(y_\alpha, x_\alpha)}{\mu_\alpha} = \frac{\omega_\alpha(y_\alpha)^{2^*/n}}{\mu_\alpha u_\alpha(y_\alpha)^{2^*/n}} \geq \omega_\alpha(y_\alpha)^{2^*/n}\ .$$

Let $\exp_{x_\alpha}$ be the exponential map at x_α with respect to the metric g. For $x \in B(0,2)$, we define

$$\hat{g}_\alpha(x) = (\exp_{y_\alpha}^* g_\alpha)(u_\alpha(y_\alpha)^{-2^*/n} x)\ .$$

and

$$v_\alpha(x) = u_\alpha(y_\alpha)^{-1} u_\alpha(\exp_{y_\alpha}(u_\alpha(y_\alpha)^{-2^*/n} x))$$

We claim that the sequence $(v_\alpha)_\alpha$ is uniformly bounded on $B(0,2)$ for $\alpha > 0$ large. In fact, there exists a constant $c > 0$, independent of α, such that for any $x \in B(0,2)$ and $\alpha > 0$ large, one has

$$d_g(x_\alpha, \exp_{y_\alpha}(u_\alpha(y_\alpha)^{-2^*/n} x)) \geq d_g(x_\alpha, y_\alpha) - d_g(y_\alpha, \exp_{y_\alpha}(u_\alpha(y_\alpha)^{-2^*/n} x))$$

$$= d_g(x_\alpha, y_\alpha) - 2u_\alpha(y_\alpha)^{-2^*/n} = (1 - 2\omega_\alpha(y_\alpha)^{-2^*/n}) d_g(x_\alpha, y_\alpha) \ .$$

Since $\omega_\alpha(y_\alpha) \to +\infty$ as $\alpha \to +\infty$, for $\alpha > 0$ large, one has

$$d_g(x_\alpha, \exp_{y_\alpha}(u_\alpha(y_\alpha)^{-2^*/n} x)) \geq \frac{1}{2} d_g(x_\alpha, y_\alpha) \ . \tag{15}$$

Hence,

$$v_\alpha(x) = u_\alpha(y_\alpha)^{-1} u_\alpha(\exp_{y_\alpha}(u_\alpha(y_\alpha)^{-2^*/n} x))$$

$$\leq 2^{n/2^*} d_g(x_\alpha, y_\alpha)^{-n/2^*} u_\alpha(y_\alpha)^{-1} \omega_\alpha(\exp_{y_\alpha}(u_\alpha(y_\alpha)^{-2^*/n} x))$$

$$\leq 2^{n/2^*} d_g(x_\alpha, y_\alpha)^{-n/2^*} u_\alpha(y_\alpha)^{-1} \omega_\alpha(y_\alpha) = 2^{n/2^*},$$

so that

$$||v_\alpha||_{L^\infty(B(0,2))} \leq 2^{n/2^*} \ . \tag{16}$$

On the other hand, v_α satisfies

$$-\Delta_{\hat{g}_\alpha} v_\alpha + B_\alpha v_\alpha = \lambda_\alpha v_\alpha^{2^*-1} \ \text{ in } \ B(0,2)$$

for a certain constant $B_\alpha > 0$, so that

$$- \Delta_{\hat{g}_\alpha} v_\alpha \le \lambda_\alpha v_\alpha^{2^*-1} \ \text{ in } \ B(0,2) \ . \tag{17}$$

Note also that

$$\hat{g}_\alpha \to \xi \ \text{ in } \ C^0(\overline{B(0,2)}),$$

so that there exist constants $\gamma, c_0 > 0$ such that

$$\hat{g}_\alpha \ge \gamma \xi \ \text{ in } \ \Omega, \tag{18}$$

in the bilinear forms sense, and

$$||(\hat{g}_\alpha)_{ij}||_{C^0(\overline{B(0,2)})} \le c_0 \tag{19}$$

for $\alpha > 0$ large. Thanks to (16), (18) and (19), the De Giorgi-Nash-Moser iterative scheme can be applied to (17), so that

$$v_\alpha(0) \le \sup_{B(0,1)} v_\alpha(x) \le c \int_{B(0,2)} v_\alpha^{2^*} \, dv_{\hat{g}_\alpha} = c \int_{B_{g_\alpha}(y_\alpha, 2u_\alpha(y_\alpha)^{-2^*/n})} u_\alpha^{2^*} \, dv_{g_\alpha}$$

for some constant $c > 0$ depending only on γ and c_0. Since $v_\alpha(0) = 1$, the desired contradiction is then obtained by showing that the right-hand side integral converges to 0 as $\alpha \to +\infty$. By the step 3, it is sufficient then to show that

$$B_{g_\alpha}(y_\alpha, 2u_\alpha(y_\alpha)^{-2^*/n}) \cap B_{g_\alpha}(x_\alpha, R\mu_\alpha) = \emptyset \ .$$

Since $g_\alpha \to g$ in C^0 and M is compact, there exists a constant $c > 0$, independent of α, such that $d_{g_\alpha} \ge c d_g$ for $\alpha > 0$ large. Then, the assertion above follows directly from

$$d_{g_\alpha}(x_\alpha, y_\alpha) u_\alpha(y_\alpha)^{2^*/n} \ge c d_g(x_\alpha, y_\alpha) u_\alpha(y_\alpha)^{2^*/n}$$

$$= c w_\alpha(y_\alpha)^{2^*/n} \ge 2 + R u_\alpha(y_\alpha)^{2^*/n} \mu_\alpha = 2 + R u_\alpha(y_\alpha)^{2^*/n} ||u_\alpha||_\infty^{-2^*/n},$$

which clearly holds for $\alpha > 0$ large, since $w_\alpha(y_\alpha) \to +\infty$. ■

Step 5: For each $\delta > 0$ small, one has

$$\lim_{\alpha \to +\infty} \frac{\int_{M \setminus B_g(x_0,\delta)} u_\alpha^2 \, dv_g}{\int_M u_\alpha^2 \, dv_g} = 0 \ . \tag{20}$$

Proof: First, by Hölder's inequality,

$$\int_{M \setminus B_g(x_0,\delta)} u_\alpha^2 \, dv_g \leq \sup_{M \setminus B_g(x_0,\delta)} u_\alpha \int_M u_\alpha \, dv_g \leq v_g(M)^{1/2} \sup_{M \setminus B_g(x_0,\delta)} u_\alpha \left(\int_M u_\alpha^2 \, dv_g \right)^{1/2} .$$

By the step 2, the C^0-convergence of g_α, (11) and (12), we can perform a De Giorgi-Nash-Moser iterative scheme in (E_α) and find a constant $c_1, c_2 > 0$, depending only on γ, c_0 and δ, such that

$$\sup_{M \setminus B_g(x_0,\delta)} u_\alpha \leq c_1 \int_M u_\alpha \, dv_{g_\alpha} \leq c_2 \int_M u_\alpha \, dv_g$$

for $\alpha > 0$ large. From two inequalities above and (E_α), one finds

$$\int_{M \setminus B_g(x_0,\delta)} u_\alpha^2 \, dv_g \leq c \left(\int_M u_\alpha^2 \, dv_g \right)^{1/2} \int_M u_\alpha^{2^*-1} \, dv_g \tag{21}$$

for α large. Now we analyze two situations. If $n = 4$, then

$$\frac{\int_{M \setminus B_g(x_0,\delta)} u_\alpha^2 \, dv_g}{\int_M u_\alpha^2 \, dv_g} \leq c||u_\alpha||_2 \to 0,$$

since $2^* - 1 = 2$. Else, if $n > 4$, then $2^* - 1 > 2$. In this case, applying a Hölder type inequality and using that $u_\alpha \in \Lambda_\alpha$, one arrives at

$$\frac{\int_{M \setminus B_g(x_0,\delta)} u_\alpha^2 \, dv_g}{\int_M u_\alpha^2 \, dv_g} \leq c||u_\alpha||_2^{(n-4)/2} \to 0 \ . \quad ■$$

Step 6: Here is the final argument. Assume that g_α converges to g in the C^2-topology. Thus, we have

$$\liminf_{\alpha \to +\infty} \mathrm{inj}_{g_\alpha}(M) > 0,$$

where $\mathrm{inj}_{g_\alpha}(M)$ denotes the injectivity radius of (M, g_α). So, there exists $\delta > 0$ small enough, independent of α, such that $B_{g_\alpha}(x_\alpha, \delta)$ is a geodesic ball for all $\alpha > 0$ large. Moreover, if $\exp_{x_\alpha, g_\alpha}$ denote the exponential map

at x_α with respect to the metric g_α, then $\exp_{x_\alpha,g_\alpha} \circ \exp^{-1}_{x_0,g}$ converges to the identity map $id : B(0,\delta) \to \mathbb{R}^n$ in the C^3-topology. For each $x \in B(0,\delta)$, we set

$$h_\alpha(x) = \exp^*_{x_\alpha,g_\alpha} g_\alpha(x)$$

and

$$v_\alpha(x) = u_\alpha(\exp_{x_\alpha,g_\alpha}(x)) \ .$$

Let $\eta \in C_0^\infty(B(0,\delta))$ be such that $\eta = 1$ on $B(0,\frac{\delta}{2})$ and $|\nabla\eta| \le c\delta^{-1}$. In the sequel, c denotes a positive constant, independent of α and δ. From the Euclidean L^2-Sobolev inequality, one has

$$\left(\int_{B(0,\delta)} (\eta v_\alpha)^{2^*} \, dv_\xi\right)^{2/2^*} \le K(n,2)^2 \int_{B(0,\delta)} |\nabla\left(\eta v_\alpha\right)|^2 \, dv_\xi \ . \tag{22}$$

As easily one checks,

$$\int_{B(0,\delta)} |\nabla(\eta v_\alpha)|^2 \, dv_\xi \le \int_{B(0,\delta)} \eta^2 v_\alpha \Delta v_\alpha \, dv_\xi + c\delta^{-2} \int_{B(0,\delta)\setminus B(0,\frac{\delta}{2})} v_\alpha^2 \, dv_\xi \ .$$

We also have

$$-\Delta v_\alpha = -\Delta_{h_\alpha} v_\alpha + (h_\alpha^{ij} - \delta_{ij})\partial_{ij} v_\alpha - h_\alpha^{ij}\Gamma(h_\alpha)_{ij}^k \partial_k v_\alpha \, ,$$

where δ_{ij} is the Kronecker's symbol and $\Gamma(h_\alpha)_{ij}^k$ denotes the Christoffel's symbols of the Levi-Civita's connection associated to the metric h_α. This gives

$$\int_{B(0,\delta)} |\nabla(\eta v_\alpha)|^2 \, dv_\xi \le -\int_{B(0,\delta)} \eta^2 v_\alpha \Delta_{h_\alpha} v_\alpha \, dv_\xi + c\delta^{-2} \int_{B(0,\delta)\setminus B(0,\frac{\delta}{2})} v_\alpha^2 \, dv_\xi$$

$$+ \int_{B(0,\delta)} \eta^2 v_\alpha (h_\alpha^{ij} - \delta_{ij})\partial_{ij} v_\alpha \, dv_\xi - \int_{B(0,\delta)} \eta^2 v_\alpha h_\alpha^{ij} \Gamma(h_\alpha)_{ij}^k \partial_k v_\alpha \, dv_\xi,$$

so that integrating by parts, using (E_α) and $\lambda_\alpha < K(n,2)^{-2}$, we get

$$\int_{B(0,\delta)} |\nabla(\eta v_\alpha)|^2 \, dv_\xi \le K(n,2)^{-2} \int_{B(0,\delta)} \eta^2 v_\alpha^{2^*} \, dv_\xi - (B_0(2,g) + \varepsilon_0) K(n,2)^{-2} \int_{B(0,\delta)} (\eta v_\alpha)^2 \, dv_\xi$$

$$+c\delta^{-2}\int_{B(0,\delta)\backslash B(0,\frac{\delta}{2})} v_\alpha^2\, dv_\xi - \int_{B(0,\delta)} \eta^2 (h_\alpha^{ij} - \delta_{ij})\partial_i v_\alpha \partial_j v_\alpha\, dv_\xi$$

$$+\frac{1}{2}\int_{B(0,\delta)} (\partial_k h_\alpha^{ij}\Gamma(h_\alpha)_{ij}^k + \partial_{ij}h_\alpha^{ij})(\eta v_\alpha)^2\, dv_\xi\ .$$

From (22), one then obtains

$$(B_0(2,g)+\varepsilon_0)\int_{B(0,\delta)} (\eta v_\alpha)^2\, dv_\xi \le \int_{B(0,\delta)} \eta^2 v_\alpha^{2^*}\, dv_\xi - \left(\int_{B(0,\delta)} (\eta v_\alpha)^{2^*}\, dv_\xi\right)^{2/2^*}$$

$$+c\delta^{-2}\int_{B(0,\delta)\backslash B(0,\frac{\delta}{2})} v_\alpha^2\, dv_\xi + \frac{K(n,2)^2}{2}\int_{B(0,\delta)} \partial_k (h_\alpha^{ij}\Gamma(h_\alpha)_{ij}^k + \partial_{ij}h_\alpha^{ij})(\eta v_\alpha)^2\, dv_\xi$$

$$-K(n,2)^2\int_{B(0,\delta)} \eta^2 (h_\alpha^{ij} - \delta_{ij})\partial_i v_\alpha \partial_j v_\alpha\, dv_\xi\ .$$

Dividing both sides by $K(n,2)^2\int_{B(0,\delta)} v_\alpha^2\, dv_\xi$ and letting $\alpha \to +\infty$, we find

$$(B_0(2,g)+\varepsilon_0)K(n,2)^{-2} \le K(n,2)^{-2}\limsup_{\alpha\to+\infty}\frac{A_\alpha}{\int_{B(0,\delta)} v_\alpha^2\, dv_\xi}$$

$$+\frac{1}{2}\limsup_{\alpha\to+\infty}\frac{B_\alpha}{\int_{B(0,\delta)} v_\alpha^2\, dv_\xi} + \limsup_{\alpha\to+\infty}\frac{C_\alpha}{\int_{B(0,\delta)} v_\alpha^2\, dv_\xi}, \tag{23}$$

where

$$A_\alpha = \int_{B(0,\delta)} \eta^2 v_\alpha^{2^*}\, dv_\xi - \left(\int_{B(0,\delta)} (\eta v_\alpha)^{2^*}\, dv_\xi\right)^{\frac{2}{2^*}},$$

$$B_\alpha = \int_{B(0,\delta)} (\partial_k (h_\alpha^{ij}\Gamma(h_\alpha)_{ij}^k) + \partial_{ij}h_\alpha^{ij})(\eta v_\alpha)^2\, dv_\xi$$

and

$$C_\alpha = \left|\int_{B(0,\delta)} \eta^2 (h_\alpha^{ij} - \delta_{ij})\partial_i v_\alpha \partial_j v_\alpha\, dv_\xi\right|\ .$$

A simple computation, using the convergence $g_\alpha \to g$ in the C^2-topology, gives

$$\lim_{\alpha\to+\infty}(\partial_k(h_\alpha^{ij}\Gamma(h_\alpha)_{ij}^k)+\partial_{ij}h_\alpha^{ij})(0)=\frac{1}{3}Scal_g(x_0),$$

so that, with the step 5,

$$\limsup_{\alpha\to+\infty}\frac{B_\alpha}{\int_{B(0,\delta)}v_\alpha^2\,dv_\xi}=\frac{1}{3}Scal_g(x_0)+\varepsilon_\delta, \tag{24}$$

where $\varepsilon_\delta\to 0$ as $\delta\to 0$. Using again the convergence in the C^2-topology together with some computations, as done in [5], one finds

$$\limsup_{\alpha\to+\infty}\frac{A_\alpha}{\int_{B(0,\delta)}v_\alpha^2\,dv_\xi}\le\frac{n-4}{12(n-1)}K(n,2)^2Scal_g(x_0)+\varepsilon_\delta \tag{25}$$

and

$$\limsup_{\alpha\to+\infty}\frac{C_\alpha}{\int_{B(0,\delta)}v_\alpha^2\,dv_\xi}\le\varepsilon_\delta\ . \tag{26}$$

Putting (24), (25) and (26) into (23), we obtain, for any $\delta>0$ small enough,

$$(B_0(2,g)+\varepsilon_0)K(n,2)^{-2}\le\frac{n-2}{4(n-1)}Scal_g(x_0)+\varepsilon_\delta\ . \tag{27}$$

Letting $\varepsilon_\delta\to 0$ as $\delta\to 0$ in (27), we arrive at the desired contradiction, since for $n\ge 4$ we have

$$(B_0(2,g)+\varepsilon_0)K(n,2)^{-2}>\frac{n-2}{4(n-1)}Scal_g(x_0)\ .$$

The C^2-topology is sharp as shows the following counter-example. Let (M,g) be a smooth compact Riemannian manifold of dimension $n\ge 4$. Consider a sequence $(f_\alpha)_\alpha\subset C^\infty(M)$ of positive functions converging to the constant function $f_0=1$ in $L^p(M)$, $p>n$, such that $\max_M f_\alpha\to+\infty$. Let $u_\alpha\in C^\infty(M)$, $u_\alpha>0$, be the unique solution of

$$-\frac{4(n-1)}{n-2}\Delta_g u+u=f_\alpha\ .$$

From the classical elliptic L^p theory, it follows that $(u_\alpha)_\alpha$ is bounded in $H_2^p(M)$, where $H_2^p(M)$ stands for the second order L^p-Sobolev space on M, so that u_α converges to u_0 in $C^{1,\beta}(M)$ for some $0<\beta<1$. Moreover, $u_0=1$, since f_α converges to 1 in $L^p(M)$ and the constant function 1 is the unique solution of the

limit problem. Therefore, $g_\alpha = u_\alpha^{2^*-2} g$ is a smooth Riemannian metric converging to g in the $C^{1,\beta}$-topology. Note also that there exists a constant $c > 0$, independent of α, such that

$$Scal_{g_\alpha} = (-\frac{4(n-1)}{n-2}\Delta_g u_\alpha + Scal_g u_\alpha) u_\alpha^{1-2^*} \geq f_\alpha u_\alpha^{1-2^*} - c u_\alpha^{2-2^*},$$

so that $\max_M Scal_{g_\alpha} \to +\infty$, where $Scal_g$ denotes the scalar curvature of the metric g. On the other hand, for $n \geq 4$, we have the well-known lower bound (see [10])

$$B_0(2, g_\alpha) \geq \frac{n-2}{4(n-1)} K(n,2)^2 \max_M Scal_{g_\alpha},$$

so that $B_0(2, g_\alpha) \to +\infty$. In particular, $B_0(2, g_\alpha) \not\to B_0(2, g)$.

■

3 Proof of Theorem 1.2

Part of the proof in the case $1 < p < 2$ follows a similar outline to the proof of Theorem 1.1. The proof in the case $p = 1$ is inspired in [8]. So, we will present a resumed proof, emphasizing the essential points. Consider initially a sequence $(g_\alpha)_\alpha \subset \mathcal{M}_2$ converging to a smooth metric g in the C^0-topology. Similarly to the previous proof, the convergence in the C^2-topology will be used only in the last step. As before, we suppose, by contradiction, that the continuity fails, so that there exists $\varepsilon_0 > 0$ such that

$$|B_0(p, g_\alpha) - B_0(p, g)| > \varepsilon_0$$

for infinitely many α. Then, at least, one of the situations holds:

$$B_0(p, g) - B_0(p, g_\alpha) > \varepsilon_0 \quad \text{or} \quad B_0(p, g_\alpha) - B_0(p, g) > \varepsilon_0$$

for infinitely many α. If the first alternative holds, then for any $u \in H_1^p(M)$,

$$\left(\int_M |u|^{p^*} \, dv_{g_\alpha}\right)^{p/p^*} \leq K(n,p)^p \int_M |\nabla_{g_\alpha} u|^p \, dv_{g_\alpha} + (B_0(p,g) - \varepsilon_0) \int_M |u|^p \, dv_{g_\alpha},$$

so that, letting $\alpha \to +\infty$, we obtain the first contradiction. Suppose then that $B_0(p, g) + \varepsilon_0 < B_0(p, g_\alpha)$ for infinitely many α. For each $\alpha > 0$, we consider the functional

$$J_{\alpha,p}(u) = \int_M |\nabla_{g_\alpha} u|^p \, dv_{g_\alpha} + (B_0(p,g) + \varepsilon_0) K(n,p)^{-p} \int_M |u|^p \, dv_{g_\alpha}$$

defined on $\Lambda_{\alpha,p} = \{u \in H_1^p(M) : \int_M |u|^{p^*} \, dv_{g_\alpha} = 1\}$. From the definition of $B_0(p, g_\alpha)$, it follows that

$$\lambda_{\alpha,p} := \inf_{u \in \Lambda_{\alpha,p}} J_{\alpha,p}(u) < K(n,p)^{-p} \ .$$

Assume first $1 < p < 2$. In this case, the inequality above combined with the classical elliptic theory of quasilinear elliptic operators imply the existence of a positive minimizer $u_\alpha \in \Lambda_{\alpha,p}$ for $\lambda_{\alpha,p}$ with $u_\alpha \in C^1(M)$. The Euler-Lagrange equation for u_α is then

$$-\Delta_{p,g_\alpha} u_\alpha + (B_0(p,g) + \varepsilon_0) K(n,p)^{-p} u_\alpha^{p-1} = \lambda_{\alpha,p} u_\alpha^{p^*-1} \ . \tag{$E_{\alpha,p}$}$$

Here, $\Delta_{p,g_\alpha} u = \mathrm{div}_{g_\alpha}(|\nabla_{g_\alpha} u|^{p-2} \nabla_{g_\alpha} u)$ denotes the p-Laplacian operator associated to the metric g_α. By the C^0-convergence of g_α, there exist constants $\gamma_1, \gamma_2 > 0$ such that $\gamma_1 g \leq g_\alpha \leq \gamma_2 g$ for $\alpha > 0$ large. This fact combined with $J_{\alpha,p}(u_\alpha) < K(n,p)^{-p}$ imply that $(u_\alpha)_\alpha$ is bounded in $H_1^p(M)$, so that $u_\alpha \rightharpoonup u$ in $H_1^p(M)$, $u \geq 0$, and

$$\int_M u_\alpha^q \, dv_{g_\alpha} \to \int_M u^q \, dv_g,$$

for any $1 \leq q < p^*$, as $\alpha \to +\infty$, up to a subsequence. Let $\lambda_{\alpha,p} \to \lambda_p$. Since $g_\alpha \to g$ in C^0, letting $\alpha \to +\infty$ in ($E_{\alpha,p}$) and using measure theory standard arguments, we find that u satisfies in the weak sense

$$-\Delta_{p,g} u + (B_0(p,g) + \varepsilon_0) K(n,p)^{-p} u^{p-1} = \lambda_p u^{p^*-1} \ . \tag{E_p}$$

If $u \not\equiv 0$, from ($J^p_{g,opt}$), (E_p) and $0 \leq \lambda_p \leq K(n,p)^{-p}$, one obtains $\int_M u^{p^*} \, dv_g > 1$, and this contradicts

$$\int_M u^{p^*} dv_g \leq \liminf_{\alpha \to +\infty} \int_M u_\alpha^{p^*} \, dv_{g_\alpha} = 1 \ .$$

Assume then that $u \equiv 0$ on M. Arguing of similar manner to the case $p = 2$, one gets $\lambda_{\alpha,p} \to K(n,p)^{-p}$. The proofs of the steps from 1 to 5, in the $p = 2$ case, relied strongly on local Hölder estimates of weak solutions of elliptic equations and on De Giorgi-Nash-Moser type iterative schemes. These tools are also

valid in the quasi-linear elliptic context, we refer to [13] and [14] for results in the quasi-linear elliptic theory. For equations as above, involving a family of p-Laplacian divergence type operators associated to g_α, $\alpha > 0$, such tools require only C^0-convergence of g_α. In fact, one needs only constants $\gamma, c_0 > 0$, independent of α, such that $g_\alpha \geq \gamma\xi$ and $||(g_\alpha)_{ij}||_{C^0} \leq c_0$ for $\alpha > 0$ large. So, the steps from 1 to 5 extend readily to $1 < p < 2$, we refer to [6] for more details. Therefore, for $1 < p < 2$, these steps take the following form:

We say that $x \in M$ is a point of concentration of $(u_\alpha)_\alpha$ if, for any $\delta > 0$,

$$\limsup_{\alpha \to +\infty} \int_{B_g(x,\delta)} u_\alpha^{p^*} \, dv_{g_\alpha} > 0 \ .$$

Step 1: The sequence $(u_\alpha)_\alpha$ possesses exactly one point of concentration x_0, up to a subsequence.

Step 2: Let $x_0 \in M$ be the unique point of concentration of $(u_\alpha)_\alpha$. Then,

$$\lim_{\alpha \to +\infty} u_\alpha = 0 \ \text{ in } \ C^0_{loc}(M \setminus \{x_0\}) \ . \tag{28}$$

Step 3: For each $R > 0$, one has

$$\lim_{\alpha \to +\infty} \int_{B_{g_\alpha}(x_\alpha, R\mu_\alpha)} u_\alpha^{p^*} \, dv_{g_\alpha} = 1 - \varepsilon_R \tag{29}$$

where $\mu_\alpha = ||u_\alpha||_\infty^{-p^*/n}$ and $\varepsilon = \varepsilon_R \to 0$ as $R \to +\infty$.

Step 4: There exists a constant $c > 0$, independent of α, such that

$$d_g(x, x_\alpha)^{n/p^*} u_\alpha(x) \leq c$$

for all $x \in M$ and $\alpha > 0$ large.

Step 5: For each $\delta > 0$ small, one has

$$\lim_{\alpha \to +\infty} \frac{\int_{M \setminus B_g(x_0,\delta)} u_\alpha^p \, dv_g}{\int_M u_\alpha^p \, dv_g} = 0 \ . \tag{30}$$

Step 6: Here, we assume that g_α converges to g in the C^2-topology. This convergence implies that

$$\liminf_{\alpha\to+\infty} \mathrm{inj}_{g_\alpha}(M) > 0 \ .$$

Thus, there exists $\delta > 0$ small enough and independent of α such that $B_{g_\alpha}(x_\alpha, \delta)$ is a geodesic ball for all $\alpha > 0$ large. In addition, $\exp_{x_\alpha, g_\alpha} \circ \exp^{-1}_{x_0, g}$ converges to the identity map $id : B(0, \delta) \to \mathbb{R}^n$ in the C^3-topology. Consider a smooth function η such that $0 \leq \eta \leq 1$, $\eta = 1$ in $(0, \delta)$, $\eta = 0$ in $(2\delta, +\infty)$ and $|\nabla \eta| \leq c/\delta$ for some constant $c > 0$ independent of δ. Define $\eta_\alpha(x) = \eta(d_g(x, x_\alpha))$. In what follows, several possibly different positive constants, independent of δ and α, will be denoted by c. Since $x_\alpha \to x_0$ and $g_\alpha \to g$ in the C^2-topology, the Cartan expansion of g_α in a normal coordinates system gives for $\alpha > 0$ large,

$$(1 - c d_{g_\alpha}(x, x_\alpha)^2) dv_{g_\alpha} \leq dv_\xi \leq (1 + c d_{g_\alpha}(x, x_\alpha)^2) dv_{g_\alpha} \ . \tag{31}$$

and

$$|\nabla(\eta_\alpha u_\alpha)|^p(x) \leq |\nabla_{g_\alpha}(\eta_\alpha u_\alpha)|^p(x)(1 + c d_{g_\alpha}(x, x_\alpha)^2) \tag{32}$$

Clearly, (31) gives

$$\int_{B_g(x_\alpha, 2\delta)} (\eta_\alpha u_\alpha)^{p^*} \, dv_\xi \geq 1 - \int_{M \setminus B_g(x_\alpha, \delta)} u_\alpha^{p^*} \, dv_{g_\alpha} - c \int_{B_g(x_\alpha, 2\delta)} u_\alpha^{p^*} d_{g_\alpha}(x, x_\alpha)^2 \, dv_{g_\alpha} \ . \tag{33}$$

By the step 2,

$$\int_{M \setminus B_g(x_\alpha, \delta)} u_\alpha^{p^*} \, dv_{g_\alpha} = o(||u_\alpha||_p^p),$$

and, by the step 4,

$$\int_{B_g(x_\alpha, 2\delta)} u_\alpha^{p^*} d_{g_\alpha}(x, x_\alpha)^2 \, dv_{g_\alpha} \leq c\delta^{2-p} ||u_\alpha||_p^p \ .$$

So, (33) yields

$$\left(\int_{B_g(x_\alpha, 2\delta)} (\eta_\alpha u_\alpha)^{p^*} \, dv_\xi \right)^{p/p^*} \geq 1 - o(||u_\alpha||_p^p) - c\delta^{2-p} ||u_\alpha||_p^p \ . \tag{34}$$

By (31) and (32), we also have

$$K(n,p)^p \int_{B_g(x_\alpha,2\delta)} |\nabla(\eta_\alpha u_\alpha)|^p \, dv_\xi \leq K(n,p)^p \int_{B_g(x_\alpha,2\delta)} |\nabla_{g_\alpha}(\eta_\alpha u_\alpha)|^p \, dv_{g_\alpha} \tag{35}$$

$$+c \int_{B_g(x_\alpha,2\delta)} |\nabla_{g_\alpha}(\eta_\alpha u_\alpha)|^p d_{g_\alpha}(x,x_\alpha)^2 \, dv_{g_\alpha} \ .$$

Independently, using that $J_{\alpha,p}(u_\alpha) = \lambda_{\alpha,p}$, $u_\alpha \in \Lambda_{\alpha,p}$ and $\lambda_{\alpha,p} < K(n,p)^{-p}$, one obtains

$$K(n,p)^p \int_{B_g(x_\alpha,2\delta)} |\nabla_{g_\alpha}(\eta_\alpha u_\alpha)|^p \, dv_{g_\alpha} \leq 1 - \int_{M\setminus B_g(x_\alpha,\delta)} u_\alpha^{p^*} \, dv_{g_\alpha} - (B_0(p,g)+\varepsilon_0) \int_{B_g(x_\alpha,\delta)} u_\alpha^p \, dv_{g_\alpha}$$

$$+c\delta^{-p} \int_{M\setminus B_g(x_\alpha,\delta)} u_\alpha^p \, dv_{g_\alpha} + c \int_{M\setminus B_g(x_\alpha,\delta)} |\nabla_{g_\alpha} u_\alpha|^p \, dv_{g_\alpha} \ .$$

In order to estimate the remaining integrals, consider a smooth function ζ such that $0 \leq \zeta \leq 1$, $\zeta = 0$ in $(0,\delta)$, $\zeta = 1$ in $(\delta,+\infty)$ and define $\zeta_\alpha(x) = \zeta(d_g(x,x_\alpha))$. Taking $\zeta_\alpha^p u_\alpha$ as a test function in (E_α), integrating by parts, using Young's inequality, one finds

$$\int_M \zeta_\alpha^p |\nabla_{g_\alpha} u_\alpha|^p \, dv_{g_\alpha} \leq \int_M \zeta_\alpha^p u_\alpha^{p^*} \, dv_{g_\alpha} + c \int_{M\setminus B_g(x_\alpha,\delta/2)} u_\alpha^p \, dv_{g_\alpha}$$

$$\leq \int_{M\setminus B_g(x_\alpha,\delta/2)} u_\alpha^{p^*} \, dv_{g_\alpha} + c \int_{M\setminus B_g(x_\alpha,\delta/2)} u_\alpha^p \, dv_{g_\alpha} = o(||u_\alpha||_p^p),$$

so that

$$\int_{M\setminus B_g(x_\alpha,\delta)} |\nabla_{g_\alpha} u_\alpha|^p \, dv_{g_\alpha} = o(||u_\alpha||_p^p) \ . \tag{36}$$

Thus,

$$K(n,p)^p \int_{B_g(x_\alpha,2\delta)} |\nabla_{g_\alpha}(\eta_\alpha u_\alpha)|^p \, dv_{g_\alpha} \leq 1 - (B_0(p,g)+\varepsilon_0) \int_{B_g(x_\alpha,\delta)} u_\alpha^p \, dv_{g_\alpha} + o(||u_\alpha||_p^p) \tag{37}$$

Taking now $\eta_\alpha^p u_\alpha d_{g_\alpha}(x,x_\alpha)^2$ as a test function in (E_α), where η_α is given in the beginning of this step, integrating by parts and again using Young's inequality, one obtains

$$\int_{B_g(x_\alpha,2\delta)} \eta_\alpha^p |\nabla_{g_\alpha} u_\alpha|^p d_{g_\alpha}(x,x_\alpha)^2 \, dv_{g_\alpha} \leq \int_{B_g(x_\alpha,2\delta)} u_\alpha^{p^*} d_{g_\alpha}(x,x_\alpha)^2 \, dv_{g_\alpha} + c \int_{M\setminus B_g(x_\alpha,\delta)} u_\alpha |\nabla_{g_\alpha} u_\alpha|^{p-1} \, dv_{g_\alpha}$$

$$+c\int_{B_g(x_\alpha,2\delta)} d_{g_\alpha}(x,x_\alpha)^{2-p}\eta_\alpha u_\alpha \eta_\alpha^{p-1}|\nabla_{g_\alpha}u_\alpha|^{p-1}d_{g_\alpha}(x,x_\alpha)^{p-1}\; dv_{g_\alpha}$$

$$\leq \int_{B_g(x_\alpha,2\delta)} u_\alpha^{p^*} d_{g_\alpha}(x,x_\alpha)^2\; dv_{g_\alpha} + c\int_{M\setminus B_g(x_\alpha,\delta)} u_\alpha^p\; dv_{g_\alpha} + c\int_{M\setminus B_g(x_\alpha,\delta)} |\nabla_{g_\alpha}u_\alpha|^p\; dv_{g_\alpha}$$

$$+\frac{1}{2}\int_{B_g(x_\alpha,2\delta)} \eta_\alpha^p|\nabla_{g_\alpha}u_\alpha|^p d_{g_\alpha}(x,x_\alpha)^2\; dv_{g_\alpha} + c\int_{B_g(x_\alpha,2\delta)} d_{g_\alpha}(x,x_\alpha)^{2-p}u_\alpha^p\; dv_{g_\alpha},$$

so that

$$\int_{B_g(x_\alpha,2\delta)} |\nabla_{g_\alpha}(\eta_\alpha u_\alpha)|^p d_{g_\alpha}(x,x_\alpha)^2\; dv_{g_\alpha} \leq c\delta^{2-p}||u_\alpha||_p^p + o(||u_\alpha||_p^p)\;. \tag{38}$$

Joining (34)-(38) and the Euclidean Sobolev inequality

$$\left(\int_{B_g(x_\alpha,2\delta)} (\eta_\alpha u_\alpha)^{p^*}\; dv_\xi\right)^{p/p^*} \leq K(n,p)^p\int_{B_g(x_\alpha,2\delta)} |\nabla(\eta_\alpha u_\alpha)|^p\; dv_\xi,$$

we obtain

$$(B_0(p,g)+\varepsilon_0)\int_{B_g(x_\alpha,\delta)} u_\alpha^p\; dv_{g_\alpha} \leq c\delta^{2-p}||u_\alpha||_p^p + o(||u_\alpha||_p^p)\;.$$

Dividing both sides of this inequality by $||u_\alpha||_p^p$, letting $\alpha\to+\infty$ and again using the step 5, one finds

$$(B_0(p,g)+\varepsilon_0)\leq c\delta^{2-p}$$

for all $\delta>0$ small. This is clearly a contradiction and prove the continuity of $B_0(p,g)$ on g for $1<p<2$. If $p=1$, we fix a sequence $(p_\alpha)_\alpha\subset(1,2)$, $p_\alpha\to 1$, and, for each $\alpha>0$, take a minimizer $u_\alpha\in\Lambda_{\alpha,p_\alpha}$ of J_{α,p_α}. Since g_α converges to g in the C^2-topology, using some ideas of this proof and a key result due to Druet in [8], it follows that $(u_\alpha)_\alpha$ converges uniformly as $\alpha\to+\infty$. But this leads directly to a contradiction, since the limit of $(u_\alpha)_\alpha$ is an extremal function associated to $B_0(1,g)+\varepsilon_0$. ∎

Acknowledgments: The first author was partially supported by Fapemig.

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